Name (IN CAPITALS): **Version #1**

Instructor: Gottfried Leibniz

Math 10550 Exam 2
Oct. 12, 2023.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 10 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
11.	
12.	
13.	
14.	
Total	

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Multiple Choice	________
11.	________
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14.	________
Total	________

Multiple Choice

1.(6pts) Find y', if

$$x^3 - 2x^2y^2 = 1.$$

Using implicit differentiation, we take derivatives of both sides to get $3x^2 - 2(2x^2yy' + 2xy^2) = 0$, or $3x^2 - 4xy^2 - 4x^2yy' = 0$. Solving for y' we get $4x^2yy' = -4xy^2 + 3x^2$, so $y' = \frac{-4xy^2+3x^2}{4x^2y}$.

(a) $\dfrac{3x^2 - 4xy^2}{4x^2y}$ (b) $\dfrac{3x^2}{4x^2y}$ (c) $\dfrac{3x^2 - 4xy^2}{4x^2}$

(d) $\dfrac{1}{3x^2 - 4xy^2}$ (e) $3x^2 - 4xy^2$

2.(6pts) Compute the tangent line to the curve given by the equation $x^2 - 4y^2 = 5$ at the point $(3, -1)$

The slope of the tangent line is the derivative of y at $x = 3$, so we differentiate implicitly to obtain $2x - 8yy' = 0$, or $y' = \frac{x}{4y}$. Plugging in the point $(3, -1)$ yields $y' = -\frac{3}{4}$. Using the equation for a line at a point, we get $y = -\frac{3}{4}(x - 3) + -1$, or $y = -\frac{3}{4}x + \frac{5}{4}$.

(a) $y = -\dfrac{3}{4}x + \dfrac{5}{4}$ (b) $y = \dfrac{3}{4}x - \dfrac{5}{4}$

(c) $y = -\dfrac{3}{2}x - \dfrac{11}{2}$ (d) $y = -\dfrac{3}{2}x - \dfrac{7}{2}$

(e) The tangent line does not exist.

3.(6pts) The height of a ball thrown straight upwards on the planet Astrid is given as

$$h(t) = -t^2 - 2t + 15$$

feet above the surface at time t(measured in second), $0 \leq t \leq t_i$, where t_i is the time at which the ball hits the surface. What is the speed of the ball at the moment of impact? (i.e. with what speed does the ball hit the surface?)

To find out the value of t_i, we find where $h(t) = 0$. Setting $h(t) = -t^2 - 2t + 15 = 0$, we obtain by factoring $(t+5)(t-3) = 0$. Since we take $t \geq 0$, this moment of impact occurs at $t_i = 3$ seconds. To find the speed at this moment, we need the absolute value of the derivative at $t = 3$. $h'(t) = -2t - 2$, so $|h'(3)| = |-8| = 8$ft./sec.

(a) 8 ft./sec. (b) 12 ft./sec. (c) 3 ft./sec. (d) 5 ft./sec. (e) 10 ft./sec.

4.(6pts) A 13 ft ladder is leaning against the side of a building when its base begins to slide away from the building. By the time the base is 12 ft from the building, the base is moving at a rate of 2 ft/s. How fast is the top of the ladder sliding down the wall at this moment?

Drawing a picture is helpful, with a right triangle of hypotenuse length 13, horizontal side x and vertical side y. With this, we know that $\frac{dx}{dt} = 2$, and we are looking for $\frac{dy}{dt}$ when $x = 12$. To find a relationship between what we know and what we need to know, we relate x and y by the Pythagorean Theorem, i.e., $x^2 + y^2 = 13^2$. When $x = 12$, we have $144 + y^2 = 169$, or $y = 5$. Differentiating, we obtain $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, or $\frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y} = -\frac{12(2)}{5} = -\frac{24}{5}$. Interpreting this as a speed, we obtain $\frac{48}{10}$ ft/s.

(a) $\frac{48}{10}$ ft/s (b) $\frac{5}{6}$ ft/s (c) $\frac{48}{25}$ ft/s (d) 24 ft/s (e) $\frac{1}{8}$ ft/s

5.(6pts) Find the linearization of $f(x) = \sqrt{17 - x^2}$ at $a = -1$ to estimate the value of $f\left(-\dfrac{1}{2}\right) = \sqrt{16.75}$.

The linearization of f is the equation for its tangent line, in this case to the point at $a = -1$. We compute $f'(x) = \frac{1}{2}\frac{-2x}{\sqrt{17-x^2}} = \frac{-x}{\sqrt{17-x^2}}$, yielding $f'(-1) = \frac{1}{4}$. The equation for the tangent line is then $L(x) = \frac{1}{4}(x+1) + f(-1) = \frac{1}{4}x + \frac{17}{4}$. $f(-\frac{1}{2})$ can then be approximated by $L(-\frac{1}{2}) = -\frac{1}{8} + \frac{17}{4} = \frac{33}{8}$.

(a) $\dfrac{33}{8}$ (b) $\dfrac{17}{4}$ (c) $\dfrac{31}{8}$ (d) $\dfrac{15}{4}$ (e) $\dfrac{9}{2}$

6.(6pts) Suppose that $f(x)$ is continuous and differentiable for all real numbers. If $2 \leq f'(x) \leq 3$ and $f(1) = 5$, what is the largest possible value of $f(3)$?

Since f is continuous and differentiable everywhere, it is continuous on $[1,3]$ and differentiable on $(1,3)$, so by the Mean Value Theorem, there is a c in $(1,3)$ so that $f'(c) = \frac{f(3)-f(1)}{3-1} = \frac{f(3)-5}{2}$. We also know that $2 \leq f'(x) \leq 3$ for all x, so the inequality must also hold true for $f'(c)$. Hence, $2 \leq f'(c) \leq 3$. Substituting $f'(c)$ as above shows that $2 \leq \frac{f(3)-5}{2} \leq 3$, so $4 \leq f(3) - 5 \leq 6$, or $9 \leq f(3) \leq 11$, so the largest possible value of $f(3)$ is 11.

(a) 11 (b) 3 (c) 6 (d) 9 (e) 15

7.(6pts) Let

$$f(x) = \frac{x}{x^2+1}.$$

Which of the following statements is true about $f(x)$?

To learn more about the local maxima and minima of f, we find its critical points and apply the first derivative test. $f'(x) = \frac{(x^2+1)-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$. f' exists everywhere, so its critical points occur when $f'(x) = 0$, or when $x = -1, 1$. Thus our intervals of interest are $(-\infty, -1), (-1, 1)$, and $(1, \infty)$. We can compute $f'(-2) < 0, f'(0) > 0$, and $f'(2) < 0$. This tells us we are decreasing on the first and third intervals while increasing on the second. Switching from decreasing to increasing at $x = -1$ yields a local minimum there, while the switch from increasing to decreasing at $x = 1$ produces a local maximum.

(a) There is a local minimum at $x = -1$ and a local maximum at $x = 1$

(b) There is a local minimum at $x = -\dfrac{1}{3}$ and a local maximum at $x = \dfrac{1}{3}$

(c) There is a local maximum at $x = -\dfrac{1}{3}$ and a local minimum at $x = \dfrac{1}{3}$

(d) There is a local maximum at $x = -1$ and a local minimum at $x = 1$

(e) There are no local minima and no local maxima on the graph of this function.

8.(6pts) Let $f(x) = x^4 - 2x^3 - 12x^2 + 2023x + 10550$. Which of the following is true? (Note: "Concave up/down on the interval (a, b)" should be interpreted as "concave up/down on the entire interval (a, b)")

To find where the graph of f is concave up or down, we need to examine the second derivative. With $f(x)$ given, $f'(x) = 4x^3-6x^2-24x+2023$ and $f''(x) = 12x^2-12x-24 = 12(x-2)(x+1)$, so $f''(x) = 0$ if $x = 2$ or $x = -1$. Hence, we are interested in $(-\infty, -1)$, $(-1, 2)$ and $(2, \infty)$. Since, $f''(-2) > 0\,, f''(0) < 0\,, f''(3) > 0$, we know that the graph of f is concave up on $(-\infty, -1)$ and $(2, \infty)$ and concave down on $(-1, 2)$.

(a) The graph of f is concave up on the intervals $(-\infty, -1)$ and $(2, \infty)$.

(b) The graph of f is concave up on the interval $(-1, 2)$.

(c) The graph of f is concave up on the interval $(-2, 1)$.

(d) The graph of f is concave down on the intervals $(-\infty, 1)$ and $(2, \infty)$.

(e) The graph of f is concave down on the intervals $(-\infty, -2)$ and $(-1, \infty)$.

9.(6pts) How many inflection points does the curve $y = \sin(x) + x^2$ have on the interval $0 < x < 4\pi$?

To see how many inflection points the curve $y = \sin(x) + x^2$ has, we need to see where the function changes from concave up to concave down or the other way around. Such points occur when the second derivative is zero (since there are no points where it fails to exist and f is continuous there) and the first derivative changes sign across that point. If $y = \sin(x)+x^2$, then $y' = \cos(x) + 2x$, and $y'' = -\sin(x) + 2 = 2 - \sin(x)$. Since $\sin(x) \leq 1 < 2$ for all x, $y'' = 2 - \sin(x) > 0$ for all x, so the graph of y has no inflection points.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

10.(6pts) Which of the following statements is false?

Statement (a) is false. For example, consider $f(x) = \begin{cases} -x^2 & x \neq 0 \\ 5 & x = 0 \end{cases}$ It clearly has a maximum of 5 on $[0, 1]$, but is not continuous on that interval. Statement (b) is true because concave down means the second derivative is negative, which tells us that the first derivative is continually decreasing. This means the linearization always overestimates. Statement (c) is true by the Extreme Value Theorem. Statement (d) is true by the Mean Value Theorem. Statement (e) is true because a derivative of zero for all numbers means there is never any change of the function. Thus it is constant for all numbers.

(a) If f is not continuous on an interval $[a, b]$, then f does not attain an absolute maximum on that interval (that is, there is no c in $[a, b]$ with $f(c) \geq f(x)$ for all x in $[a, b]$).

(b) If f is concave down on $(-\infty, \infty)$, then any linear approximation will be an overestimate. (that is, $L(x) \geq f(x)$ for any x in $(-\infty, \infty)$ when $L(x)$ is the linear approximation to $f(x)$ at a point $x = a$).

(c) A function that is continuous on a closed interval $[a, b]$ attains an absolute maximum and an absolute minimum on that interval (that is, there is a c in $[a, b]$ with $f(c) \geq f(x)$ for all x in $[a, b]$, and a d in $[a, b]$ with $f(d) \leq f(x)$ for all x in $[a, b]$).

(d) If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) where the slope of the tangent line at c is the same as the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

(e) If $f'(x) = 0$ for all x in $(-\infty, \infty)$, then $f(x)$ must be a constant function. That is, $f(x) = c$ for every x in $(-\infty, \infty)$ for some real number c.

Partial Credit

For full credit on partial credit problems, make sure you justify your answers.

11.(12pts) The position of a particle, moving along a horizontal axis is given by

$$s(t) = t^3 - 3t^2 - 9t + 10, \qquad \text{for } 0 \le t \le 10,$$

where distance is measured in feet and time in seconds.

(Show all of your work below for credit)

(a) Find the velocity, $v(t)$, and acceleration, $a(t)$, of the particle for $0 \le t \le 10$.

The velocity of the particle is $v(t) = s'(t) = 3t^2 - 6t - 9$, and the acceleration is $a(t) = v'(t) = 6t - 6$.

(b) What is the total distance travelled by the particle between $t = 0$ seconds and $t = 10$ seconds?

Note $s(10) = 620$.

To find the total distance, we first need to see where the particle changes direction. This corresponds to when its velocity is 0. Hence, setting $v(t) = 0$, we get $3(t^2 - 2t - 3) = 3(t-3)(t+1) = 0$. Since $t \ge 0$, we only accept $t = 3$ seconds. Hence, the total distance travelled would be $|s(0) - s(3)| + |s(3) - s(10)| = |10 - (-17)| + |-17 - 620| = 664$ feet.

(c) On what time intervals (for for $0 \le t \le 10$) is the particle speeding up and and on what intervals is it slowing down?

The particle is speeding up when $v(t)$ and $a(t)$ have the same sign, and slowing down when they have opposite signs. Notice that $a(t) = 0$ when $t = 1$ second. Combined with the solution for $v(t) = 0$ in part b, we are interested in the intervals $(0,1)$, $(1,3)$ and $(3,10)$. On the interval $(0,1)$, since $v(0.5) < 0$ and $a(0.5) < 0$, $v(t)$ and $a(t)$ are both negative. A similar reasoning shows that on $(1,3)$, $v(t) < 0$ and $a(t) > 0$, while on $(3,10)$, both $v(t)$ and $a(t)$ are positive. Hence, the particle is speeding up on $(0,1)$ and $(3,10)$ and slowing down on $(1,3)$.

12.(14pts) The function $D(t)$ shown below is a model for the depth of water at the center of the pond beside Mathtown during the month of August. Time (t) is measured in days after Aug 01 and the depth of the water (D) is measured in feet.

$$D(t) = 1 + \frac{10t}{t^2+9}, \qquad \text{for } 1 \le t \le 30,$$

(a) Calculate $D'(t)$.

Using the quotient rule, we have $D'(t) = \frac{(t^2+9)(10)-10t(2t)}{(t^2+9)^2} = \frac{10t^2+90-20t^2}{(t^2+9)^2} = \frac{90-10t^2}{(t^2+9)^2} = \frac{10(3-t)(3+t)}{(t^2+9)^2}$.

(b) Find the critical numbers of $D(t)$.

Since D' is defined for all t (the denominator can never be zero), we only need to find where $D'(t) = 0$. Solving that for t, we get $t = 3\,, -3$. Since $t \ge 1$, we see that $t = 3$ is the only critical number of $D(t)$.

(c) What theorem guarantees that $D(t)$ must have an absolute maximum value and an absolute minimum value on the interval $1 \le x \le 30$? Why does the theorem apply to the function $D(t)$ given above on the interval $[1, 30]$?

Since f is continuous on $[1, 30]$, the Extreme Value Theorem ensures us to find an absolute maximum and minimum of $D(t)$ on the given interval.

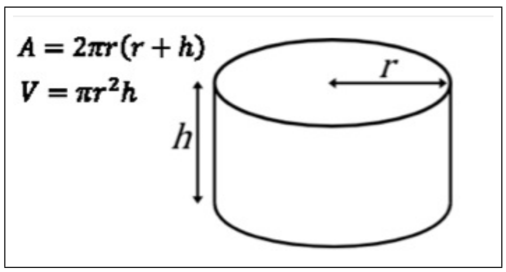
(d) Find the absolute maximum and abs olute minimum value of $D(t)$ (given above) on the interval $1 \le t \le 30$ showing your work.

Note: If you are having difficulty comparing values of $D(t)$ it may help to keep in mind that $D(t) = 1 + \dfrac{10}{t+\frac{9}{t}}$ when $t > 0$.

To find the absolute maximum and minimum of $D(t)$, we need to check for the values $D(1)\,, D(30)$ and $D(3)$. We have $D(1) = 2\,, D(3) = 1 + \frac{5}{3} = \frac{8}{3}$ and $D(30) = 1 + \frac{10}{30+\frac{9}{30}} = 1 + \frac{10}{30+\frac{3}{10}} = 1 + \frac{10}{\frac{303}{10}} = 1 + \frac{100}{303} = \frac{403}{303}$, so the absolute maximum of $D(t)$ on the given interval is $\frac{8}{3}$ ft, and the minimum is $\frac{403}{303}$ ft.

13.(12pts) Gas is pumped into a cylindrical shaped balloon in preparation for the Arthur's day parade in Dublin. The height of the balloon is twice the radius throughout the process. Gas is pumped in at a steady rate of 3 cubic feet per second. Find the rate of change of the surface area of the balloon when the radius is 3 feet.

Since the height is twice the radius in the whole process, $h = 2r$ in the formulas below. This allows us to rewrite the formulas as $A = 2\pi r(3r) = 6\pi r^2$ and $V = \pi r^2(2r) = 2\pi r^3$. We are given $\frac{dV}{dt} = 3$ cubic feet per second, and we need to calculate $\frac{dA}{dt}$ when $r = 3$ ft. To do that, we need to calculate $\frac{dr}{dt}$ when $r = 3$ ft. This can be done by differentiating the formula for the volume with respect to time. Doing that, we will get $\frac{dV}{dt} = 6\pi r^2 \frac{dr}{dt}$. Plugging in what we know, we get $3 = 6\pi(3)^2 \frac{dr}{dt}$, or $\frac{dr}{dt} = \frac{1}{18\pi}$ ft/s. To find $\frac{dA}{dt}$, we differentiate the equation $A = 6\pi r^2$ with respect to time to get $\frac{dA}{dt} = 12\pi r \frac{dr}{dt}$. Plug in the value of $\frac{dr}{dt}$ we just found, we see that $\frac{dA}{dt} = 12\pi(3) \cdot \frac{1}{18\pi} = 2$ square feet per second.



14.(2pts) You will be awarded these two points if you write your name in CAPITALS on the front page and you mark your answers on the front page with an X through your answer choice like so: (a) (<u>not</u> an O around your answer choice) . You may also use this page for

ROUGH WORK